The geometry and combinatorics of closed geodesics on hyperbolic surfaces

Chris Arettines

CUNY Graduate Center

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Motivating Question: How are the algebraic/combinatoric properties of closed geodesics related to topological/geometric properties?

Algebraic Descriptions: Reduced cyclic words, edge-crossing sequences
Topological/Geometric properties: Minimal intersection numbers, filling property

Question: Given a gluing pattern for a polygon P representing a surface S, and an edge crossing sequence for a closed curve γ on S, can we determine:

- A configuration for γ which minimizes the number of self-intersections?
- Whether or not γ is filling?

The Combinatorial Homotopy Algorithm

Theorem

(Hass and Scott 1985) If a curve in a free homotopy class has excess self-intersection, then the curve contains a proper bigon or monogon, which can be removed via homotopy.

Idea: Encode a curve combinatorially, search this encoding for proper bigons or monogons.

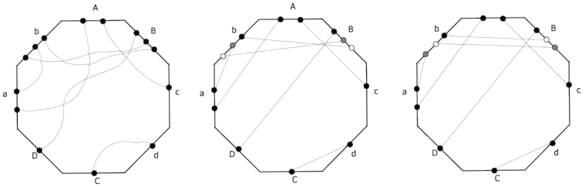


Figure: An arbitrary curve in the free homotopy class $[a^2cdB^3]$ is first straightened out into line segments, and then modified to remove bigons.

The Combinatorial Homotopy Algorithm

Proposition

The number of intersections and presence of bigons is completely encoded by the cyclic labeling of points along ∂P that the curve crosses, and the pairs of points which are connected by line segments.

Using this information, we can identify *combinatorial bigons*, which are paired sequences of segments which whose initial and terminal pairs of segments cross.

Question: Can we distinguish between proper and improper bigons just using the combinatorial information?

The Combinatorial Homotopy Algorithm

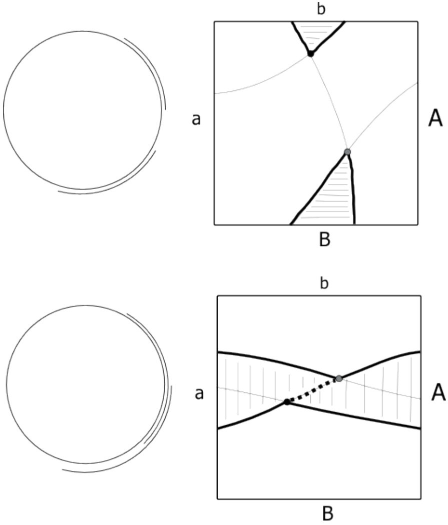


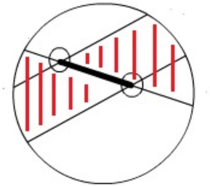
Figure: An example of a proper and improper bigon on the torus, along with schematic preimages on S^1.

Answer: Almost.

Theorem

A combinatorial bigon corresponds to an improper bigon on the surface if:

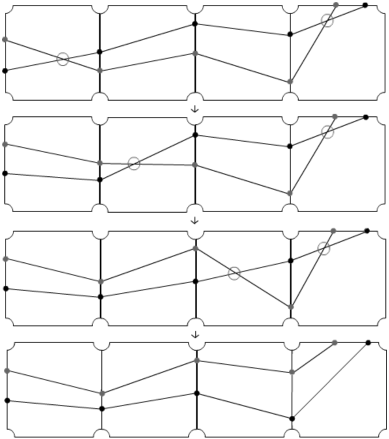
- *The two sequences of segments determining the bigon have* ≥ 2 *segments in common.*
- *The two sequences of segments determining the bigon have* 1 *segment in common, and the relative positions of the endpoints of the segments are as in the following figure.*



The Combinatorial Homotopy Algorithm

Theorem

If a combinatorial bigon is not one of the types in the previous theorem, then it is removable.



The Minimal Linking Algorithm

Definition

Two geodesics are *hyperbolically linked* if their endpoints alternate on $\partial\mathbb{H}^2$.

Definition

Two subwords of W of length one, w_j and w_k, are said to be a *short link* if the sequence of letters $w_{j-1}^{-1}, w_{k-1}^{-1}, w_j, w_k$ has no repetitions and appears in clockwise order in the labeling of the fundamental polygon.

The Minimal Linking Algorithm

Definition

Two subwords of W of length $l > 1$, $W_j = w_j w_{j+1} \dots w_{j+l-1}$ and $W_k = w_k w_{k+1} \dots w_{k+l-1}$ are said to be a *parallel long link* if:

1. $w_{j+i} = w_{k+i} \forall\ 0 < i < l - 1$
2. $w_j^{-1} \neq w_k$ and $w_{j+l-1} \neq w_{k+l-1}$
3. The two sequences of letters w_j^{-1},w_k^{-1},$w_j = w_k$ and w_{j+l-1},w_{k+l-1},$w_{j+l-2} = w_{k+l-2}$ appear in clockwise order in the labeling of the fundamental polygon.

Definition

Two subwords of W of length $l > 1, W_j = w_j w_{j+1} \dots w_{j+l-1}$ and $W_k = w_k w_{k+1} \dots w_{k+l-1}$ are said to be an *alternating long link* if W_j and $W_k{}^{-1}$ form a parallel long link.

Theorem

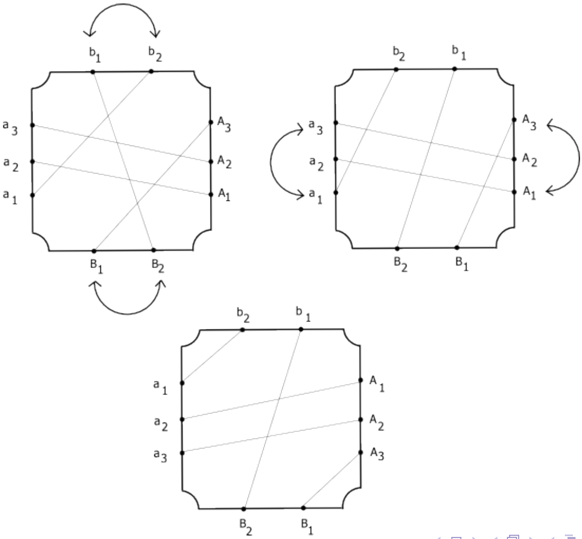
(Cohen-Lustig 1987) Intersections in a minimal representative of a primitive free homtopy class with word W are in bijection with the set of short, parallel, and alternating links in W.

The Minimal Linking Algorithm

Algorithm

1. Form the list of links uniquely determined by your input.
2. Choose a side of the polygon and list the collection of arcs entering the same edge of the polygon.
3. Choose a pair of arcs along this side and ask following: Are these arcs part of a link?
   - If not, then you know the relative positions of the endpoints are such that they do not cross.
   - If so, are these arcs part of a link that has been previously considered?
     - If not, choose the relative positions of their endpoints so that they cross. If the arcs are part of a long link, this decides the relative positions of the other endpoints involved in the link as well.
     - If so, you already know their relative positions from a previous step in the algorithm.
4. Until all pairs have been exhausted, choose a new arc and pair it with all of the arcs that have been considered up to this point, asking the question in step 3.
5. Repeat steps 2 through 4 for a new side that has not been examined until all arcs are exhausted.

Example: $A^3 b^2$



Filling algorithm

Proposition

If γ is a curve of word length n which does not fill, then there is a curve of word length $\leq 2n$ which does not intersect γ.

Question: Can we do better than this brute force method?
Answer: Yes, we can quickly compute the *relative boundary* of any curve γ.

Definition

The *essential subsurface* of curve $\gamma \subset S$ is the smallest complexity π_1-injective subsurface S' which contains γ.

Definition

The *relative boundary* of an essential subsurface $S' \subset S$ is the collection of free homotopy classes in $\pi_1(S)$ corresponding to the boundary curves of S'.

Filling algorithm

Idea: Position γ minimally using one of the earlier algorithms, and follow the relative boundary of γ to see if the boundary words are trivial.

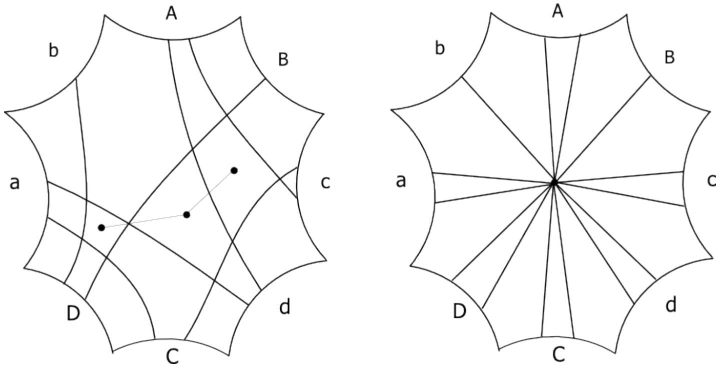


Figure: The configuration on the right is obtained by compressing all of the interior disks to points. The collection of curves on the left is filling only if the homotoped collection on the right is filling.

Filling algorithm

Definition

A *maximal linked chain* of segments $W_1, W_2, \ldots, W_k$ starting at an endpoint w of W_1 along ∂P is a sequence of segments such that:

1. Each W_i crosses each W_{i+1}
2. Each $W_j, j > 1$ has an endpoint which is cyclically closer to w in the clockwise direction than any other segment intersecting W_{j-1}
3. There is no segment intersecting W_k which has an endpoint closer to w in the clockwise direction than the endpoint of W_k closer to w in the clockwise direction.

. This closest endpoint of W_k will be called the *terminal point of the chain* and the edge the terminal point lies on will be called the *terminal edge*.

Filling algorithm

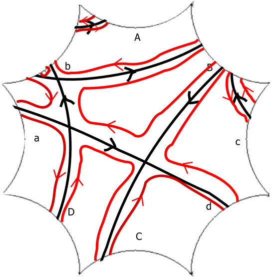
Theorem

The maximal linked chains of γ are combined in a unique way to determine the relative boundary of the essential subsurface determined by γ. If these words are all trivial, then γ is a filling curve.

Curve: AdbCBB

Boundary Word:

abbbBBADcdCB ~ BabADcdC ~ 1



Constructing filling curves

Dealing with filling curves on a surface directly can be quite intimidating...

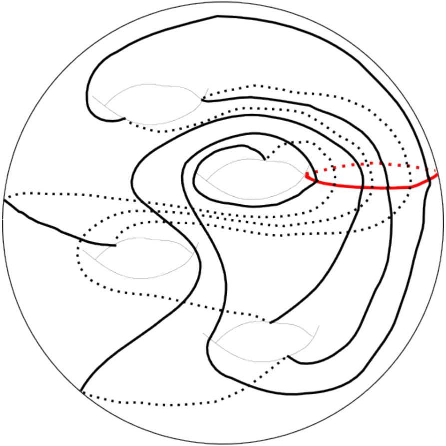


Figure: A pair of simple closed curves which fill a surface of genus 4.

Constructing filling curves

Proposition

If a collection of curves fills a surface S_g, then the collection of curves must have at least $2g - 1$ intersections.

Question: Are there filling curves with minimal possible number of intersections?

Answer: Yes, which we will see using *ribbon graphs*.

Definition

A *ribbon graph* (or *fat graph*) is a graph Γ with a chosen cyclic ordering of the half-edges at each vertex of Γ.

A 4-valent ribbon graph defines a curve on a surface with boundary, and by plugging in disks, a curve on a surface without boundary.

Constructing filling curves

Question: After we plug in disks to the surface with boundary associated to a ribbon graph, when does the core curve have minimal intersection?

Lemma

If a ribbon graph Γ is not minimal and the defined curve contains a monogon, then there is a vertex v and an oriented smooth path p of edges starting and ending at v such that:

- *Every path of edges p' starting at an intersection with p with orientation o, also intersects p at another point with orientation $-o$.*
- *If p' is a path as in the previous item, then if p'' is any path intersecting p', then p'' also intersects p.*

Lemma

If a ribbon graph Γ is not minimal and the defined curve contains a bigon, then Γ has a pair of vertices v and w connected by smooth paths p_1 and p_2 such that:

- *Each path p' intersecting p_1 or p_2, must also intersect p_1 or p_2, with the appropriate orientations at the points of intersection.*
- *If p' is a path as in the previous item, then if p'' is any path intersecting p', then p'' also intersections p.*

Constructing filling curves

Figure: The left configuration does not determine a bigon, while the right one does.

Constructing filling curves

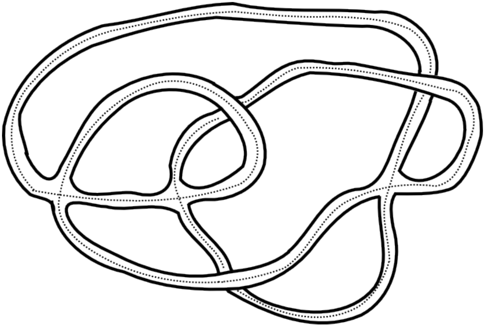
Theorem

For $g \geq 2$, there exists a curve γ with $2g - 1$ self-intersections, whose complement is a single topological disk.

Theorem

For $g \geq 2$, there exists a curve γ with $2g$ self-intersections, whose complement is a pair of topological disks.

Constructing filling curves



Constructing filling curves

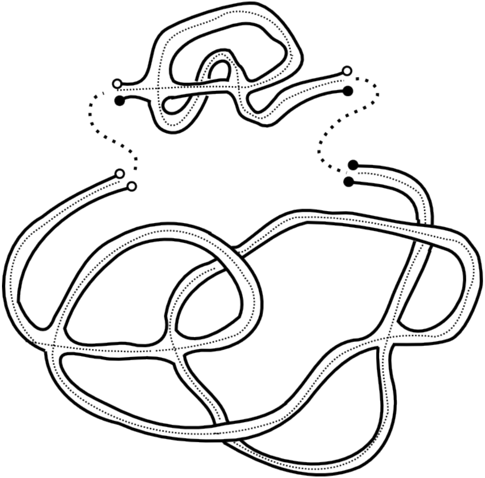


Figure: We may attach several of the pieces depicted here side by side. For each piece we attach, we increase the genus of the surface by one without increasing the number of boundary components.

Constructing filling curves

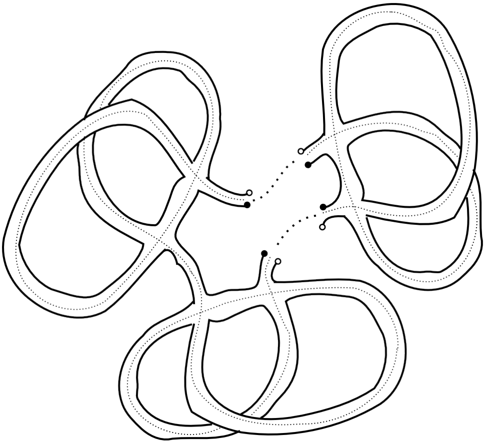


Figure: The larger piece can be glued back up to produce an example in genus 2, while additional pieces can be glued to obtain arbitrarily higher genus examples.

Constructing filling curves

Question

Can every minimally self-intersecting filling curve on a surface S_g be obtained by surgering a minimal filling curve on S_{g-1} as in the earlier figure?

Question

How many homeomorphism classes of minimally self-intersecting filling curves are there? We can obtain different homeomorphism classes by attaching the new piece to different ribbons in the diagram.

The curve to polygons map

Idea: If a curve γ fills a surface S equipped with a hyperbolic metric, then the complement of γ is a collection of hyperbolic polygons or once-punctured polygons.

If γ has no *triangular regions*, then this collection of polygons is a topological invariant, and doesn't depend on the particular hyperbolic metric.

Thus, we a map ϕ_γ from Teichmüller space into a product of configuration spaces for polygons. This map goes in reverse too: from a labeled collection of hyperbolic polygons satisfying certain edge-length and angle conditions (determined by γ), we can reconstruct the hyperbolic structure on S.

Using the polygon map

We can use ϕ_γ to obtain information about γ and the overall hyperbolic metric.

Theorem

A minimally intersecting filling curve on a surface S_g must have length at least half that of a regular right-angled $(8g-4)$-gon. This lower bound is:

$$(4g-2) \cdot cosh^{-1}\left(2 \cdot cos\left[\frac{\pi}{4g-2}\right] + 1\right)$$

Proof sketch:

- The complement of a filling curve with $2g-1$ self-intersections is an $8g-4$ sided polygon.
- For a polygon with fixed area (in this case $2\pi(2g-2)$), the regular polygon with this area minimizes the perimeter.
- The regular polygon with this area is right-angled, and the perimeter can be computed using hyperbolic trigonometry.

Angles of intersection

Question: To what extent can specific information about polygons in the complement of filling curves be used to understand the hyperbolic metric?

Question: Can angles of intersection be used as parameters for Teichmüller space?

Okumura (1996,1997) has found collections of curves whose angles of intersection determine the overall metric.

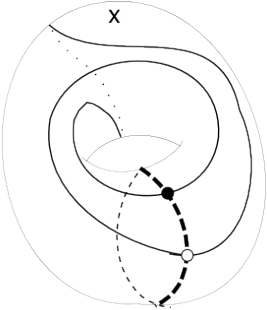
Question: Can we find smaller collections?

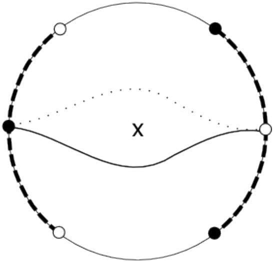
Angles of intersection

Proposition

Let α and β be two simple closed curves on the punctured torus which intersect twice, and let α' and β' be another such pair of curves. Then there is a homeomorphism of the surface ϕ which sends $\alpha \cup \beta$ to $\alpha' \cup \beta'$.

Proof sketch:





Angles of intersection

The complement of $\alpha \cup \beta$ is a quadrilateral and punctured quadrilateral with alternating angles.

Proposition

A quadrilateral has alternating angles if and only if it has alternating side lengths.

Theorem

The space of alternating quadrilaterals is homeomorphic to an open subset of $\mathbb{R}^3$ given by sending a quadrilateral $\mathcal{Q}$ to the triple (x, θ_1, θ_2), where x is the length of a side, and θ_1 and θ_2 are the alternating angles.

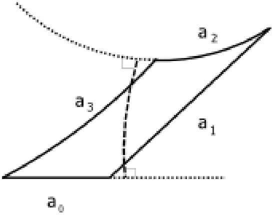
Angles of intersection

Definition

A quadrilateral whose sides are labeled $a_0, .., a_3$ with $length(a_0) = length(a_2) = x$ and $length(a_1) = length(a_3) = y$ is called a *good quadrilateral with respect to side* a_0 if the orthosegment between the two geodesics containing a_0 and a_2 is $arcsinh(\frac{1}{sinh(x)})$.

Proposition

The space of good quadrilaterals is homeomorphic to $\mathbb{R}_+ \times \mathbb{R}$.



Angles of intersection

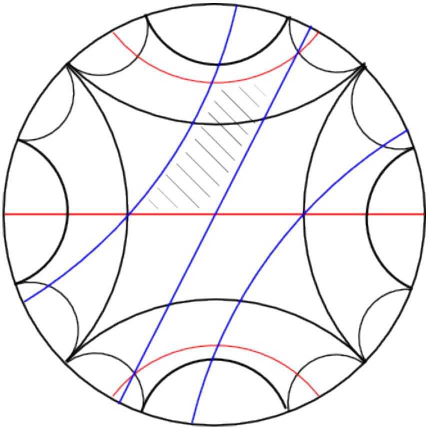


Figure: Some lifts of α and β, and the good quadrilateral they determine.

Angles of intersection

Theorem

If $\mathcal{Q}$ is good with respect to some side a_0, then $\mathcal{Q}$ determines a unique hyperbolic structure for the punctured torus.

Proof sketch: "Goodness" condition allows you to reconstruct an ideal quadrilateral fundamental domain using $\mathcal{Q}$. This determines a hyperbolic structure.

Proposition

If $\mathcal{Q}$ is a good quadrilateral with respect to side a_i, then it is also good with respect to side a_{i+1}.

Two operations on the space of good quadrilaterals

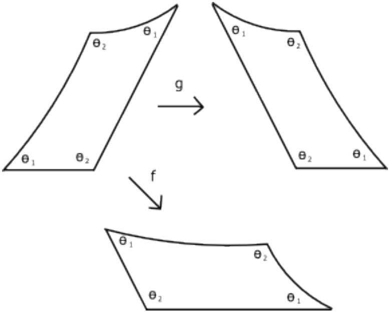


Figure: The natural geometric operations on a quadrilateral Q.

- f cyclically permutes the labels of the sides, in effect rotating the quadrilateral.
- g switches the locations of the two angles.

Angles of intersection

Theorem

Let $\mathcal{Q}$ be a good quadrilateral. Then $g \circ f(\mathcal{Q})$ is a good quadrilateral with the same angles.

This tells us that angles of intersection cannot be used as global coordinates.

Proposition

$g \circ f$ has a unique fixed point.

Theorem

Let α and β be any two simple curves which intersect minimally twice and fill on the punctured torus. Then the two angles of intersection give local parameters for Teichmüller space, except at a discrete set of points.

Future questions

Idea: Fix a topological or geometric observable. What subset of Teichmüller space preserves this observable?

If a filling curve has triangular regions, then there are several topological configurations possible for the curve.
Question: Which regions correspond to each configuration?
Answer: Unknown in general, but sometimes the empty set.

Every filling curve is associated to a unique metric which minimizes the length (Kerckhoff).
Question: If we pick a larger length, then what subset of Teichmüller space preserves this length?
Answer: A set homeomorphic to a sphere.

Future questions

What about the subset of Teichmüller space which preserves a tuple of intersection angles for a filling curve? We have just analyzed a special case of this on the punctured torus. The answer for a general curve on $S_{g,n}$ will surely be more complicated. The following heuristic observation suggests that this may be a very interesting set in some cases.

Observation

Let γ be a minimally self-intersecting filling curve on S_g, and let $\Theta = (\theta_1, ..., \theta_{2g-1})$ be a tuple of intersection angles realized in some hyperbolic metric. Then the set of Teichmüller space which preserves Θ should have dimension $4g - 5$.

The number $4g - 5$ comes up in the literature as the conjectural lower bound for the dimension of a deformation retract of moduli space (Harer, Ji-Wolpert).

Thank you

Thank you for your time and attention.