BERS EMBEDDINGS, SKINNING MAPS AND HYPERBOLIC GEOMETRY

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Based on a lecture by Dr. Yair Minsky at the Bers 100 Conference

Abstract. In the pioneering work of Thurston, we find beautiful connections between the theory of Riemann surfaces, deformations of groups, hyperbolic geometry, and the topology of 3-manifolds. Here we will attempt to survey some of the ideas surrounding Thurston's skinning map, which draw heavily from the work of Bers and many others.

1. Deformations of Kleinian Groups

Let's recall some basic definitions to get started. A *Kleinian* group is a discrete subgroup of $PSL(2, \mathbb{C})$, and a *quasi-Fuchsian* group is a Kleinian group whose limit set Λ is a Jordan curve. The story begins with the simultaneous uniformization theorem, first proved by Bers[3]:

Theorem 1.1. *Let S be a topological surface, and let X and Y be two conformal structures on S. Then there is a single quasi-Fuchsian group Γ preserving each component of the sphere minus the limit set, so that the quotient by the action is $X \sqcup Y$.*

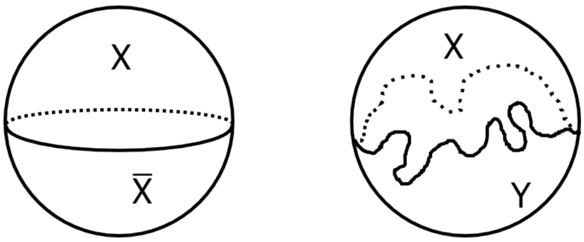

Figure 1. The limit set of a general quasi-Fuchsian group will be a complicated Jordan curve. Each of the two disjoint domains of discontinuity yield X and Y respectively when quotiented out by the group.

This theorem and the body of ideas surrounding it has many generalizations. For a given Kleinian group Γ, one can consider the space of all quasi-conformal conjugates of it, denoted $Q(\Gamma)$. The following important theorem has many ingredients, some of which can be found in [1], [4], [10], [14].

Theorem 1.2. *Let Γ be a finitely generated Kleinian group. If each component of the domain of discontinuity Ω is simply connected, then $Q(\Gamma) \simeq T(\Omega/\Gamma)$, where $T(\Omega/\Gamma)$ is the Teichmüller space of the (orbifold) Riemann surface Ω/Γ.*

Another way to say this is that quasi-conformal deformations of Γ are controlled by deformations of Fuchsian surface groups. In the special case where $\Gamma_0 \simeq \pi_1(S)$

is a Fuchsian group uniformizing a Riemann surface S, we recover the simultaneous uniformization theorem, which yields $Q(\Gamma_0) \simeq T(S) \times T(S)$. For $X \in T(S)$, we can consider the set of all quasi-Fuchsian deformations of Γ_0 in $T(S) \times T(S)$ whose first coordinate is X. This image of $(X, *)$ in $Q(\Gamma_0)$ is called a *Bers slice* B_X. It turns out that B_X has compact closure in the space of discrete and faithful representations $\Gamma_0 \to PSL(2, \mathbb{C})$. This follows from Chuckrow's theorem, which states that a limit of discrete and faithful representations of such a group Γ_0 must also be discrete and faithful (see [7]). Thurston leans on compactness theorems like this in his work on hyperbolization.

2. The Skinning Map

The final inductive step of Thurston's proof that closed, atoroidal Haken 3-manifolds admit hyperbolic structures can be described as solving a gluing problem. In this step, there are two hyperbolic 3-manifolds M_1 and M_2, each with homeomorphic boundary surfaces. One then has to find deformations of the conformal structures on these boundary surfaces so that the gluing can be achieved. This can be formulated as a fixed-point problem for a certain map on Teichmüller space, which we now describe. See [12] for a beautiful exposition of these ideas.

For simplicity, let $M \simeq \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold so that the conformal boundary $S = \Omega/\Gamma$ is connected, and that $\bar{M} = M \cup S$ is compact. Furthermore assume that the boundary is *incompressible*, that is, $\pi_1(\partial \bar{M}) \to \pi_1(M)$ is injective. With this assumption, we can consider the the cover $p : \hat{M} \to M$ corresponding to $\pi_1(\partial \bar{M}) \subset \pi_1(M)$. The hyperbolic structure on Γ pulls back to a hyperbolic structure on $\hat{M}$, and there is an associated map between the deformation spaces $p_* : Q(\Gamma) \to Q(\pi_1(\hat{M}))$. With a little bit of work, one can prove that $\pi_1(\hat{M})$ is a quasi-Fuchsian group, so we may apply the earlier deformation theorems to see that we really have a map $p_* : T(S) \to T(S) \times T(S)$. The image of a point $X \in T(S)$ is a pair (X, Y), which can be seen by looking at the geometry of the covering manifold $\hat{M}$. We define $\sigma_M(X)$ to be Y, which is the so-called skinning map for M. Intuitively, $\sigma_M(X)$ is the hyperbolic structure that you see looking into the manifold from the boundary.

Now, back to the gluing problem. The set up is that we have two hyperbolic 3-manifolds M_1 and M_2, each with homeomorphic boundary surfaces. Suppose that we are given a gluing map $f : \partial M_1 \to \partial M_2$ identifying these boundaries. When and how can we find a hyperbolic structure on the glued manifold? There cannot always be a solution: if M_1 and M_2 contain cylinders, then for certain choices of f we might form an incompressible but non-peripheral torus, which cannot exist in a hyperbolic 3-manifold. To simplify the following discussion, we assume that M_1 and M_2 are acylindrical.

If ∂M_1 has structure X, and ∂M_2 has structure Y, then in order for these structures to be compatible, we must have that $f_* \sigma_{M_1}(X) = Y$ and $f_*^{-1} \sigma_{M_2}(Y) = X$. See Figure 2 for a schematic picture. This condition is also sufficient, which can be seen using a combination theorem in the style of the Klein-Maskit combination theorems. Therefore, solving the gluing problem is equivalent to finding a fixed point for the composition of these maps on Teichmüller space. The main ingredients for a proof of this fact are the following theorems:

Theorem 2.1. *σ_M is contracting.*

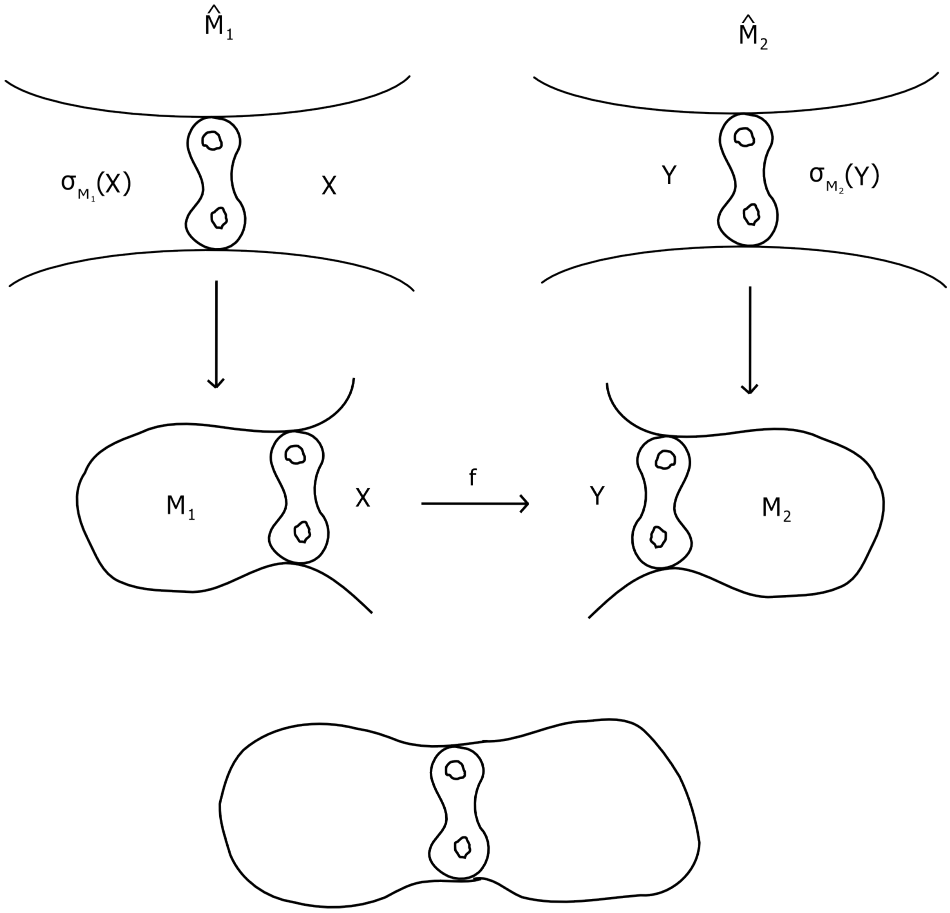

FIGURE 2. The manifold we obtain by gluing ∂M_1 to ∂M_2 via f will admit a hyperbolic structure only if the skinning images align correctly.

Theorem 2.2. *$\sigma_M(T(\partial M))$ is bounded.*

Letting $M = M_1 \sqcup M_2$, f can be thought of as a map $\partial M \to \partial M$. Then, combining the previous theorems with the fact that f_* acts by isometries, we may iterate $f_* \circ \sigma_M$ starting from any point in $T(\partial M)$ to yield a limiting hyperbolic structure, which is clearly a fixed point.

3. Quantitative Questions

There are many questions surrounding the skinning map. One can ask in particular for information about the diameter of the image $\sigma_M(T(\partial M))$, which we henceforth denote by $\text{diam}(\sigma_M)$. Information about this quantity can be used to build bi-Lipschitz models for manifolds which are often simpler to understand. See the results in [5] for details. Not surprisingly, the diameter is not uniformly bounded across all 3-manifolds.

Theorem 3.1. [8] *There exists a sequence of manifolds M_k so that diam(σ_{M_k}) $\to \infty$.*

One feature of this sequence is that that genus of the boundaries also increases without bound. Therefore, it seems reasonable to ask the following question:

Question 3.2. *Are there constants $C(g)$ so that if the genus of $\partial M = g$, then diam(σ_M) $< C(g)$?*

In certain settings, we do have more precise information about the image which supports an affirmative answer. To organize what follows, let us first consider the following, which comes from the fixed-point machinery that Thurston developed:

Theorem 3.3. *If M is an acylindrical 3-manifold with incompressible boundary, then there is a unique hyperbolic structure on M so that the boundary group is Fuchsian.*

In other words, for this very special choice of hyperbolic metric, there is a totally geodesic "mirror surface" S with structure X_0 contained in the manifold. More precisely, $\sigma_M(X_0) = \bar{X}_0$. This can be thought of as a natural "basepoint" for studying the skinning image. Results about the skinning image are often obtained by considering different ways in which X_0 could degenerate.

Let d(M) be the length of the largest collar of this totally geodesic surface.

Theorem 3.4. [8] *If M_k is a sequence of manifolds with $d(M) \to 0$, then diam(σ_M) $\to \infty$.*

A theorem of Basmajian [2] says that if ∂M has fixed genus, then there is a lower bound for d(M), so any sequence as in the theorem necessarily has $g \to \infty$. More evidence in favor of the conjecture is the following:

Theorem 3.5. [8] *Fix $k > 0$. Let M be any acylindrical 3-manifold with incompressible boundary, whose convex core M_0 has volume $< k$. Then diam σ_M is bounded.*

The proof of this fact depends on a delicate compactness argument. Morally, when vol(M_0) is bounded, you're almost looking at a finite collection of manifolds (up to sequences of manifolds converging to manifolds with cusps).

One way in which the structure X_0 can degenerate is by forcing X_0 to have a bounded length pants decomposition. The following theorem jointly due to Bromberg and Kent is a result in this direction:

Theorem 3.6. [8] *Given $\epsilon > 0, \exists \delta > 0$ so that if X_0 admits a pants decomposition whose curves have length $< \delta$, then diam(σ_M) $< \epsilon$.*

This can be thought of as handling "the worst" way in which X_0 could degenerate to the boundary of Teichmuller space, with all pants curves shrinking to zero length. Now let's consider a complementary situation where X_0 remains in a compact part of Teichmuller space, and allow the collar length d(M) to increase.

Theorem 3.7. [9] *Given g, ϵ, $\exists$ constants A, d_0 (depending on g and ϵ) so that if ∂M has genus g, and the totally geodesic surface X_0 is ϵ-thick, then if $d(M) > d_0$, diam(σ_M) $< Ae^{d(M)}$.*

The proof uses a gluing argument to build candidate hyperbolic structures that estimate the skinning map using a theorem of Tian [17]. The required control on the gluing is obtained using a type of C^∞ foliation developed by Epstein [6], where the smoothness comes from the holomorphic nature of the Bers slice.

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