Algorithmic solutions to topological questions about closed geodesics on surfaces.

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Motivating Question: How are the algebraic/combinatoric properties of closed geodesics related to topological/geometric properties?

Algebraic Descriptions: Reduced cyclic words, edge-crossing sequences
Topological/Geometric properties: Minimal intersection numbers, filling property

Question: Given a gluing pattern for a polygon P representing a surface S, and an edge crossing sequence for a closed curve γ on S, can we determine:

- A configuration for γ which minimizes the number of self-intersections?
- Whether or not γ is filling?

The Combinatorial Homotopy Algorithm

Theorem

(Hass and Scott 1985) If a curve in a free homotopy class has excess self-intersection, then the curve contains a proper bigon or monogon, which can be removed via homotopy.

Idea: Encode a curve combinatorially, search this encoding for proper bigons or monogons.

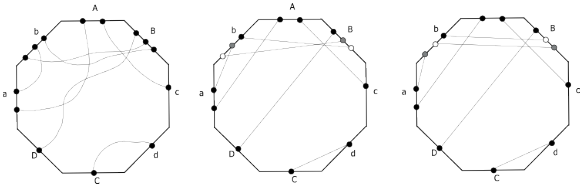


Figure: An arbitrary curve in the free homotopy class $[a^2cdB^3]$ is first straightened out into line segments, and then modified to remove bigons.

The Combinatorial Homotopy Algorithm

Proposition

The number of intersections and presence of bigons is completely encoded by the cyclic labeling of points along ∂P that the curve crosses, and the pairs of points which are connected by line segments.

Using this information, we can identify *combinatorial bigons*, which are paired sequences of segments which whose initial and terminal pairs of segments cross.

Question: Can we distinguish between proper and improper bigons just using the combinatorial information?

The Combinatorial Homotopy Algorithm

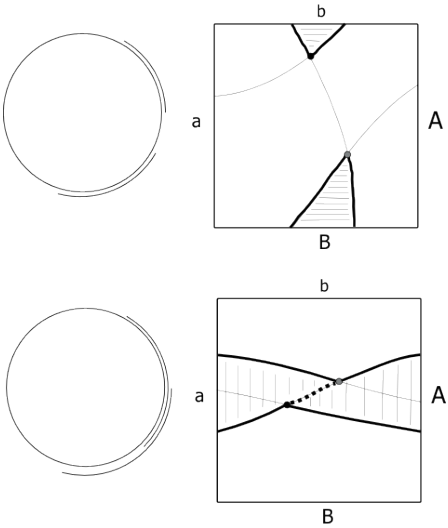


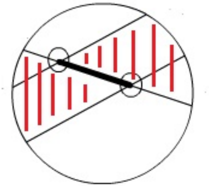
Figure: An example of a proper and improper bigon on the torus, along with schematic preimages on S^1.

Answer: Almost.

Theorem

A combinatorial bigon corresponds to an improper bigon on the surface if:

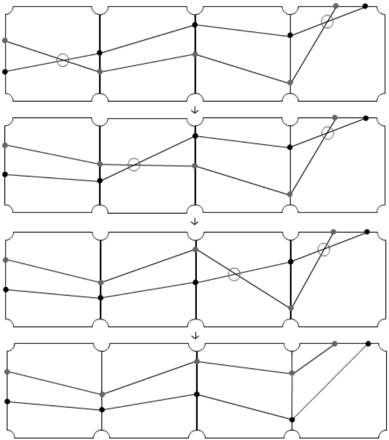
- *The two sequences of segments determining the bigon have* ≥ 2 *segments in common.*
- *The two sequences of segments determining the bigon have* 1 *segment in common, and the relative positions of the endpoints of the segments are as in the following figure.*



The Combinatorial Homotopy Algorithm

Theorem

If a combinatorial bigon is not one of the types in the previous theorem, then it is removable.



The Minimal Linking Algorithm

Definition

Two geodesics are *hyperbolically linked* if their endpoints alternate on $\partial\mathbb{H}^2$.

Definition

Two subwords of W of length one, w_j and w_k, are said to be a *short link* if the sequence of letters $w_{j-1}^{-1}, w_{k-1}^{-1}, w_j, w_k$ has no repetitions and appears in clockwise order in the labeling of the fundamental polygon.

The Minimal Linking Algorithm

Definition

Two subwords of W of length $l > 1$, $W_j = w_j w_{j+1} \ldots w_{j+l-1}$ and $W_k = w_k w_{k+1} \ldots w_{k+l-1}$ are said to be a *parallel long link* if:

1. $w_{j+i} = w_{k+i} \forall\ 0 < i < l - 1$
2. $w_j^{-1} \neq w_k$ and $w_{j+l-1} \neq w_{k+l-1}$
3. The two sequences of letters w_j^{-1},w_k^{-1},$w_j = w_k$ and w_{j+l-1},w_{k+l-1},$w_{j+l-2} = w_{k+l-2}$ appear in clockwise order in the labeling of the fundamental polygon.

Definition

Two subwords of W of length $l > 1, W_j = w_j w_{j+1} \ldots w_{j+l-1}$ and $W_k = w_k w_{k+1} \ldots w_{k+l-1}$ are said to be an *alternating long link* if W_j and $W_k{}^{-1}$ form a parallel long link.

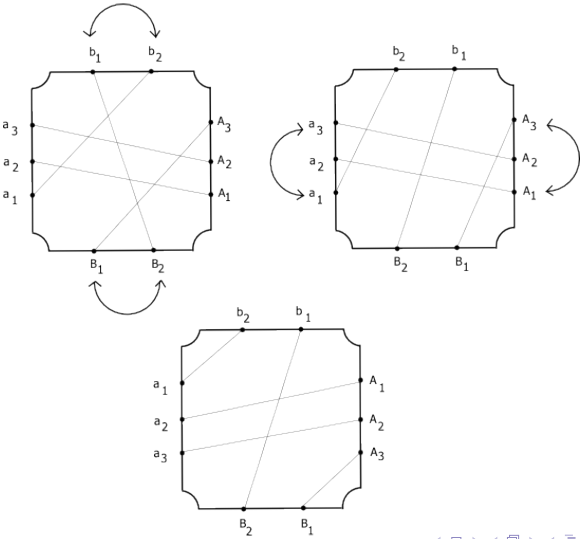
Theorem

(Cohen-Lustig 1987) Intersections in a minimal representative of a primitive free homtopy class with word W are in bijection with the set of short, parallel, and alternating links in W.

Algorithm

1. Form the list of links uniquely determined by your input.
2. Choose a side of the polygon and list the collection of arcs entering the same edge of the polygon.
3. Choose a pair of arcs along this side and ask following: Are these arcs part of a link?
   - If not, then you know the relative positions of the endpoints are such that they do not cross.
   - If so, are these arcs part of a link that has been previously considered?
     - If not, choose the relative positions of their endpoints so that they cross. If the arcs are part of a long link, this decides the relative positions of the other endpoints involved in the link as well.
     - If so, you already know their relative positions from a previous step in the algorithm.
4. Until all pairs have been exhausted, choose a new arc and pair it with all of the arcs that have been considered up to this point, asking the question in step 3.
5. Repeat steps 2 through 4 for a new side that has not been examined until all arcs are exhausted.

Example: A^3b^2



Filling algorithm

Proposition

If γ is a curve of word length n which does not fill, then there is a curve of word length $\leq 2n$ which does not intersect γ.

Question: Can we do better than this brute force method?
Answer: Yes, we can quickly compute the *relative boundary* of any curve γ.

Definition

The *essential subsurface* of curve $\gamma \subset S$ is the smallest complexity π_1-injective subsurface S' which contains γ.

Definition

The *relative boundary* of an essential subsurface $S' \subset S$ is the collection of free homotopy classes in $\pi_1(S)$ corresponding to the boundary curves of S'.

Filling algorithm

Idea: Position γ minimally using one of the earlier algorithms, and follow the relative boundary of γ to see if the boundary words are trivial.

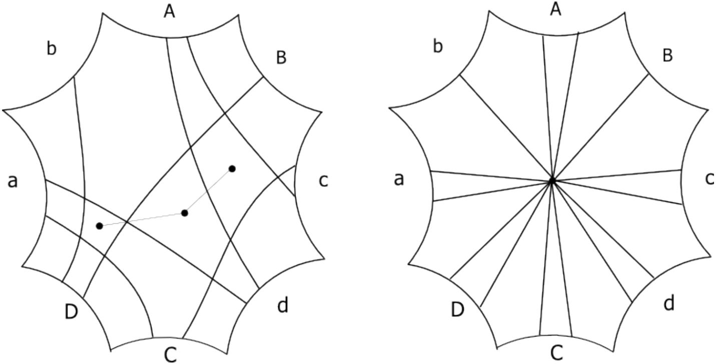


Figure: The configuration on the right is obtained by compressing all of the interior disks to points. The collection of curves on the left is filling only if the homotoped collection on the right is filling.

Filling algorithm

Definition

A *maximal linked chain* of segments $W_1, W_2, \ldots, W_k$ starting at an endpoint w of W_1 along ∂P is a sequence of segments such that:

1. Each W_i crosses each W_{i+1}
2. Each $W_j, j > 1$ has an endpoint which is cyclically closer to w in the clockwise direction than any other segment intersecting W_{j-1}
3. There is no segment intersecting W_k which has an endpoint closer to w in the clockwise direction than the endpoint of W_k closer to w in the clockwise direction.

. This closest endpoint of W_k will be called the *terminal point of the chain* and the edge the terminal point lies on will be called the *terminal edge*.

Filling algorithm

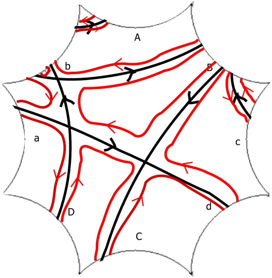
Theorem

The maximal linked chains of γ are combined in a unique way to determine the relative boundary of the essential subsurface determined by γ. If these words are all trivial, then γ is a filling curve.

Curve: AdbCBB

Boundary Word:

abbbBBADcdCB ~ BabADcdC ~ 1



Constructing filling curves

Dealing with filling curves on a surface directly can be quite intimidating...

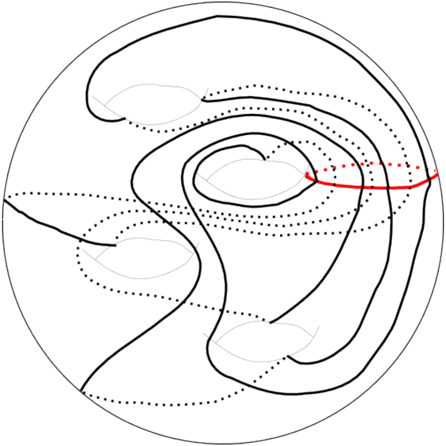


Figure: A pair of simple closed curves which fill a surface of genus 4.

Constructing filling curves

Proposition

If a collection of curves fills a surface S_g, then the collection of curves must have at least $2g - 1$ intersections.

Question: Are there filling curves with minimal possible number of intersections?

Answer: Yes, which we will see using *ribbon graphs*.

Definition

A *ribbon graph* (or *fat graph*) is a graph Γ with a chosen cyclic ordering of the half-edges at each vertex of Γ.

A 4-valent ribbon graph defines a curve on a surface with boundary, and by plugging in disks, a curve on a surface without boundary.

Constructing filling curves

Question: After we plug in disks to the surface with boundary associated to a ribbon graph, when does the core curve have minimal intersection?

Lemma

If a ribbon graph Γ is not minimal and the defined curve contains a monogon, then there is a vertex v and an oriented smooth path p of edges starting and ending at v such that:

- *Every path of edges p' starting at an intersection with p with orientation o, also intersects p at another point with orientation $-o$.*
- *If p' is a path as in the previous item, then if p'' is any path intersecting p', then p'' also intersects p.*

Lemma

If a ribbon graph Γ is not minimal and the defined curve contains a bigon, then Γ has a pair of vertices v and w connected by smooth paths p_1 and p_2 such that:

- *Each path p' intersecting p_1 or p_2, must also intersect p_1 or p_2, with the appropriate orientations at the points of intersection.*
- *If p' is a path as in the previous item, then if p'' is any path intersecting p', then p'' also intersections p.*

Constructing filling curves

Figure: The left configuration does not determine a bigon, while the right one does.

Constructing filling curves

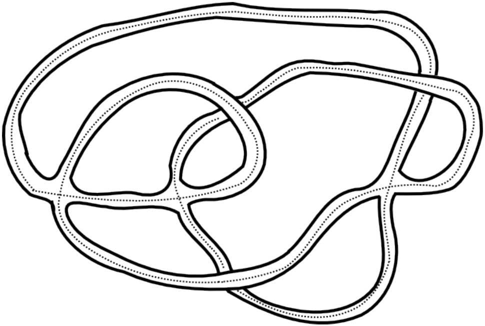
Theorem

For $g \geq 2$, there exists a curve γ with $2g - 1$ self-intersections, whose complement is a single topological disk.

Theorem

For $g \geq 2$, there exists a curve γ with $2g$ self-intersections, whose complement is a pair of topological disks.

Constructing filling curves



Constructing filling curves

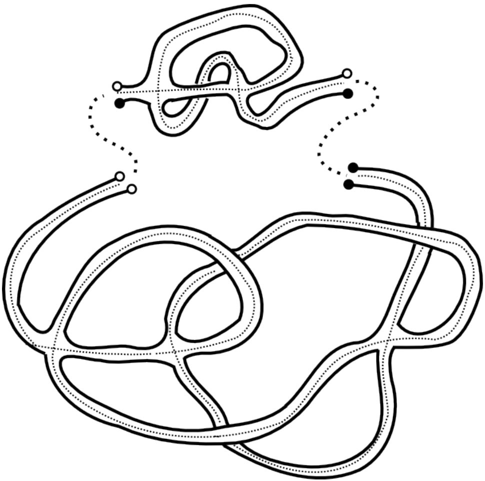


Figure: We may attach several of the pieces depicted here side by side. For each piece we attach, we increase the genus of the surface by one without increasing the number of boundary components.

Constructing filling curves

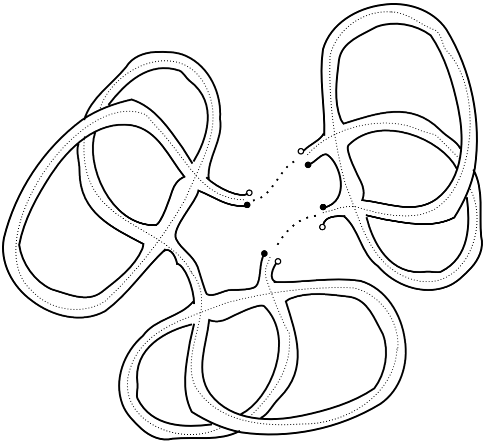


Figure: The larger piece can be glued back up to produce an example in genus 2, while additional pieces can be glued to obtain arbitrarily higher genus examples.

Constructing filling curves

Question

Can every minimally self-intersecting filling curve on a surface S_g be obtained by surgering a minimal filling curve on S_{g-1} as in the earlier figure?

Question

How many homeomorphism classes of minimally self-intersecting filling curves are there? We can obtain different homeomorphism classes by attaching the new piece to different ribbons in the diagram.

Encoding filling curves

Over the next few slides, I'll describe another way we can think about filling curves.

Observation: Let γ_i be a collection of filling curves on $S_g, g \geq 2$ whose complement is a single disk (thus with $k = 2g - 1$ total intersections). Then the ribbon graph can be homotoped to a *ribbon bouquet* with $1 + k$ bands and one vertex.

Idea: From original ribbon graph, pick a spanning tree, which has k vertices, and $k - 1$ edges.

Observation: Every ribbon bouquet can be encoded by a permutation σ_r, or equivalently, a chord diagram.

Theorem

Let $\sigma_f = (123...(2+2k))$ be the cyclic permutation shifting the labels 1 through $2+2k$. Then the number of boundary components of the ribbon bouquet encoded by σ_r is the number of cycles in $\sigma_r \circ \sigma_f$.

Encoding filling curves

Observation: Given a ribbon bouquet, we can recover a curve by choosing how to connect the bands. A choice of how to connect the bands corresponds to another chord diagram, or equivalently, a permutation σ_c.

Question: Which chord diagram choices preserve the number of boundary components?
Answer: *Tree chord diagrams* - chord diagrams whose underlying intersection graph is a tree.

So every minimal filling curve can be determined by two permutations σ_r and σ_c, each of which is a product of disjoint transpositions. How many components does this curve have? This is answered by the following:

Theorem

The number of components of a curve determined by σ_r and σ_c is half the number of cycles in $\sigma_c \circ \sigma_r$.

This framework might offer an avenue through which to assault the counting orbits question. The counting problems may be easier to tackle in this combinatorial landscape, where we can draw on results about permutations.

Thank you

Thank you for your time and attention.